Effect of Sheared Flow on the Growth Rate and Turbulence Decorrelation

Ö. D. Gürcan*
Laboratoire de Physique des Plasmas, Ecole Polytechnique, CNRS, 91128 Palaiseau Cedex, France
(Received 31 May 2012; published 12 October 2012)

The effect of a large scale flow shear on a linearly unstable turbulent system is considered. A cubic equation describing the effective growth rate is obtained, which is shown to reduce to well-known forms in weak and strong shear limits. A shear suppression rule is derived which corresponds to the point where the effective growth rate becomes negative. The effect of flow shear on nonlinear mode coupling of drift or Rossby waves is also considered, and it is shown that the resonance manifold shrinks and weakens as the vortices are sheared. This leads to a reduction of the efficiency of three-wave interactions. Tilted eddies can then only couple to the large scale sheared flows, because the resonance condition for that interaction is trivially satisfied. It is argued that this leads to absorbtion of the sheared vortices by large scale flow structures. Studying the form of the effective growth rate for weak shear, it was shown that in addition to reducing the overall growth rate, a weak flow shear also reduces the wave number where the fluctuations are most unstable.

DOI: 10.1103/PhysRevLett.109.155006 PACS numbers: 52.35.Ra, 52.35.Kt, 92.10.hf

A large number of problems in nature can be formulated in the form of free energy sources driving instabilities leading to either regular or chaotic behavior depending on the further development of the instability, such as turbulent movement of a fluid, which may happen in the form of regular patterns as in the case of Bénard cells [1], or in the form of a power law spectrum as in the case of Kolmogorov spectrum [2].

Certain turbulent systems, in particular quasi two-dimensional systems such as fluids subjected to rotation, or plasmas under the action of a strong magnetic field, are susceptible to formation of large- and meso-scale sheared flows [3]. Here we consider those systems that can develop such flow shear, and as a result are more effective in enduring large amounts of external free energy than those that cannot. In such a system, the initial instability becomes merely a catalyst that transfers the free energy from the external agent to the large scale flows, because the generated large scale flow shear reacts back on the turbulence and suppresses its own driver [4]. Systems showing this kind of behaviour include drift instabilities in fusion plasmas [5], baroclinic instabilities in geophysical fluid dynamics [6], and convection in rotating fluids [7]. The mathematical formulation described here can also be applied to systems where the flow shear is externally imposed, in which case the stability of the sheared flow itself to Kelvin-Helmholtz instability should also be considered.

While this transfer of free energy from the external agent to flow shear may lead to a reasonable quasisteady state with flows and turbulence, it may also predictably lead to repeating intermittent cycles in the manner of predator-prey oscillations between the growth of the primary instability and the growth of the secondary flow structures, and a consequent decay of the primary instabilities [5,8].

The stages of these oscillations are the initial growth, the secondary growth of the sheared flow, the suppression of the primary instability by the flow shear, and the damping of the sheared flow. The first phase is that of a trivial linear growth and is not universal as it depends on particulars of the system. The second stage is the drive of sheared flows via nonlinear stresses. In most cases, these are Reynolds stresses but could be Maxwell or kinetic stresses that can be developed depending on the problem in consideration. The fourth and last stage, the damping of the flow itself also has no universality and usually a linear effect (though higher order nonlinear effects may introduce effective damping terms on sheared flows). Here we will focus on the third stage, which is the suppression of the primary instability by the velocity shear [9] induced by large scale flow structures.

Development and further evolution of instabilities that reduce free energy sources and associated structures such as large scale flows are above all, problems of turbulent transport. As a result, the effect of large scale flow shear on turbulent decorrelation (and therefore the turbulent transport) has been studied in some detail in mainly engineering problems. The phenomenon is studied in detail for instance in the context of low to high confinement (L-H) transition in fusion plasma devices [10].

In order to study this problem we consider a system, by a generic equation of the form

$$[\partial_t + v(x)\partial_y]\tilde{\psi} + (i\omega^{(r)} - \gamma)\tilde{\psi} - \nu\nabla^2\tilde{\psi} + \tilde{v}\cdot\nabla\tilde{\psi} = 0. \quad (1)$$

Here $\tilde{\psi}$ may be a fluctuating passive scalar, or an active one such as potential vorticity that defines the velocity field itself. This generic system is taken to be linearly unstable with a given growth rate γ, has a very small dissipation νk^2 (i.e., $\gamma \gg \nu k_0^2$ where k_0 is the characteristic wave number of the instability), and a turbulence decorrelation rate $\tau_{ac} \equiv Dk^2$ representing the nonlinear term, so that the mixing length transport driven by this kind of system in its steady state would be proportional to the turbulent diffusion coefficient $D \approx \gamma/k^2$, in the absence of flow shear [11]. Note that this relation does not imply a fixed value of D but rather it gives the value at which the fluctuation level saturates in order to balance the linear growth rate.

Now introduce the so-called shearing coordinates [12]:

$$\tau = (t - t_0), \qquad x' = x, \qquad y' = y - v'x(t - t_0), \qquad (2)$$

where v' is a constant shear in the x direction of a large scale flow in the y direction that acts on turbulence [e.g., $v(x) = xv'$ in (1)]. It is clear that for the full predator-prey cycle, one cannot consider the effect of flow shear as a constant external shear. Nonetheless, this is the first thing to do in order to understand the basic mechanism of flow shear suppression of the instability.

Consider the linear response function (for $\gamma \gg \nu k^2$):

$$R_{k,0}(\tau) = \exp\{-i\omega_k\tau - \gamma_B^3\tau^3\},$$

where $\omega_k = \omega^{(r)} + i\gamma$, which may eventually depend on k_y, and $\gamma_B = (\nu k_y^2 v'^2/3)^{1/3}$ is the Biglari-Diamond-Terry [13] shear decorrelation rate (except here we rely on a model small scale dissipation ν for simplicity, whereas in the original Biglari-Diamond-Terry paper the mechanism was that of turbulent decorrelation). Note that because we can choose the initial time (t_0 above) where we start the shearing coordinate transformation, it makes sense to choose it as the initial time (say t') of the response function. As this is the linear response function, it does not include the turbulence decorrelation Dk^2.

Let us assume that in the presence of flow shear, we can define an effective growth rate $\gamma_{\rm eff}$, which is the growth rate modified by the effect of flow shear. Obviously the effect of the transformation in (2) is not to simply change the growth rate. The idea here is that it changes the response function in a complex way, but we can *fit* that response function by an exponential [i.e., $R_{\rm eff}(\tau) = e^{\gamma_{\rm eff}\tau}$ where the fitting can be made by matching the e-folding points of the two exponentials] and argue that this is the effective growth rate (note that $\gamma_{\rm eff}$ has to be strictly real). This approach is used commonly in devising turbulence closures (e.g., the eddy damped quasinormal Markovian closure [14]), where the fitting is complicated because the exact analytical form of the response function is *a priori* unknown.

Using the response function above, we can compute the effective growth rate as follows:

$$\gamma_{\rm eff}^3 = \gamma_{\rm eff}^2\gamma - \gamma_B^3, \qquad (3)$$

which gives $\gamma_{\rm eff} = \gamma$ for $\gamma_B \to 0$ and $\gamma_{\rm eff} = -\gamma_B$ for $\gamma/\gamma_B \to 0$. One interesting note here is that these two limiting values are actually on different branches of the

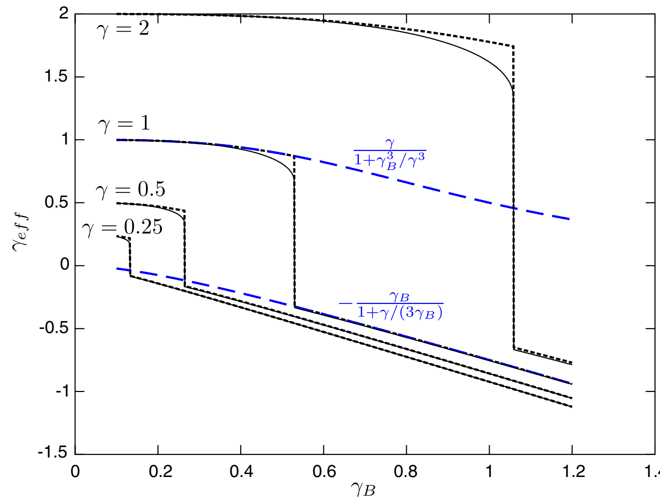

FIG. 1 (color online). The curve $\gamma_{\rm eff}$ as a function of γ_B for given values of γ, which starts at the root corresponding to $\gamma_{\rm eff} = \gamma$ and follows this root until it disappears at the critical shear value given in (6), at which point it jumps to the lower root. The weak and strong turbulence approximations given in (4) and (5) connected with the jump at the critical point (the short dashed lines) represent the full curve reasonably well. Also the continuations of the analytic formulas into irrelevant regions are shown for $\gamma = 1$ (long dashed lines, blue if in color).

solution of the cubic equation. So the effective growth rate has to *jump* from one solution to another as we increase the shear (see Fig. 1).

Considering the weak shear case, by writing the above equation as $\gamma\gamma_{\rm eff}^{-1} - \gamma_B^3\gamma_{\rm eff}^{-3} = 1$, expanding $\gamma_{\rm eff}^{-1} = \gamma_{\rm eff,0}^{-1} + \gamma_{\rm eff,1}^{-1}$ and iterating by assuming $\gamma \gg \gamma_B$, we find, $\gamma_{\rm eff,0}^{-1} = \gamma^{-1}$ and $\gamma_{\rm eff,1}^{-1} = \gamma_B^3\gamma^{-4}$ or

$$\gamma_{\rm eff} = \frac{\gamma}{(1 + \gamma^{-3}\gamma_B^3)} = \frac{\gamma}{(1 + \alpha v'^2)}, \qquad (4)$$

where $\alpha = \frac{\nu k_y^2}{3\gamma^3}$. With this reduced effective growth rate, one can compute the effective diffusion coefficient as $D_{\rm eff} = |\gamma_{\rm eff}|/k^2$ which means that the mixing length fluctuation level (i.e., ε as the intensity of the fluctuations) between the case with and without flow shear are linked as $\varepsilon = \varepsilon_0 \frac{\gamma_{\rm eff}}{\gamma}$, and for weak shear this becomes:

$$\varepsilon = \frac{\varepsilon_0}{(1 + \alpha v'^2)}.$$

Note that this is the *ad hoc* suppression formula used in a number of transport models in fusion, modeling the L-H transition [15].

In the limit of strong shear, the solution is $\gamma_{\rm eff} = -\gamma_B$. Expanding around this solution as before we find

$$\gamma_{\rm eff} = -\frac{\gamma_B}{(1 + \frac{\gamma}{3\gamma_B})}. \qquad (5)$$

Note that the weak and strong shear solutions given in (4) and (5) correspond to distinct roots of the cubic equation (3). The first root is valid until $\gamma_B^3/\gamma^3 = 4/27$ or in terms of v':

$$|v'| = \frac{2}{3}\frac{\gamma^{3/2}}{\sqrt{\nu k_y^2}}. \quad (6)$$

This is the point where the weak shear solution disappears, and the only solution is the strong shear one. This corresponds to the *shear suppression rule* [16,17], because below this critical value velocity shear only weakly affects the instability, but above it, the effective growth rate suddenly becomes negative. Note that the effective decorrelation rate is the absolute value of this. These weak and strong shear cases, seem to naturally correspond to the weak and strong nonlinear cases discussed in Ref. [18] using a self-consistent formulation of the Charney–Hasegawa–Mima system. The application of this approach to complex models is nontrivial, partly due to difficulties in estimating the model diffusivity ν, and also due to the way the shear may modify linear physics (e.g., modification of the eigenmode structure, etc.).

Physically, shear damping happens because the flow shear increases the effective wave number of the fluctuations [19,20] in the direction along the shear (i.e., perpendicular to the flow). This sends the fluctuations to regions in k space where the dissipation is important (because the dissipation goes with νk^2). As a result, we get a reduction of the growth rate for weak shearing rates and damping for strong ones, up to a Kelvin–Helmholtz instability of the sheared flow itself.

Above, we considered this effect and argued that due to the mixing length rule, as the effective growth rate decreases, the fluctuation level also decreases. It is important here to note that the mixing length rule applies in the original coordinates, and not in the shearing coordinates. In fact the turbulence is approximated to grow as $e^{\gamma_{\rm eff} t}$ in the original coordinates, which is to be balanced by the nonlinear mixing.

Shear may also eventually modify the nonlinear interactions, depending on the problem. While this does not really change the mixing length argument (because in whatever way the nonlinearity is modified, it still has to balance the effective growth rate if we have a steady state compatible with this argument), it can change spectral dynamics, spatial spreading, and pattern selection. In other words, a general mixing length rule still determines how fast the energy is extracted (by nonlinear interactions) from a most unstable mode, but it does not tell us where this extracted energy goes, and this may in principle depend on shear.

Let us consider for instance a resonance manifold given by the Manley–Rowe relations from a dispersion relation of the form $\omega_k = k_y/(1 + k^2)$. This is the dispersion relation for drift waves in a plasma [21], or Rossby waves in a quasigeostrophic fluid [22]. This problem has a nonlinear interaction coefficient which can be written as (e.g., see Ref. [23] for the notation): $M_{kpq} \equiv \hat{\mathbf{z}} \times \mathbf{p} \cdot \mathbf{q}(p^2 - q^2)/(1 + k^2)$.

The effect of a constant shear, which can be described by the shearing coordinate transformation (2) is to make k_x and therefore k^2 time dependent in the shearing frame. This modifies the resonance manifold as can be seen in Fig. 2. Please note that this is the condition the *waves* has to satisfy as their k_x is being changed by the sheared flow. The convention is that for three positive amplitudes M_{kpq} positive means k loses its energy to p and q.

It is commonly argued that in plasmas and quasigeostrophic fluids, weaknesses in resonant interactions (i.e., that the manifold is small and the interaction coefficient vanishes on a large portion of that manifold) is one of the primary reasons that favors formation of large scale flow structures [24]. In other words, as the resonance manifold is weakened the only interaction that remains is via the large scale flow that satisfies the Manley–Rowe relations in a trivial way.

It is therefore interesting to see that the action of flow shear indeed weakens the resonance manifold in time; hence, a time averaged *effective* resonant manifold also gets weaker. This leads to the turbulence being forced to couple even more to the large scale flow. In other words, while at the beginning of the shearing transformation the turbulence may have been coupling to other modes, as the

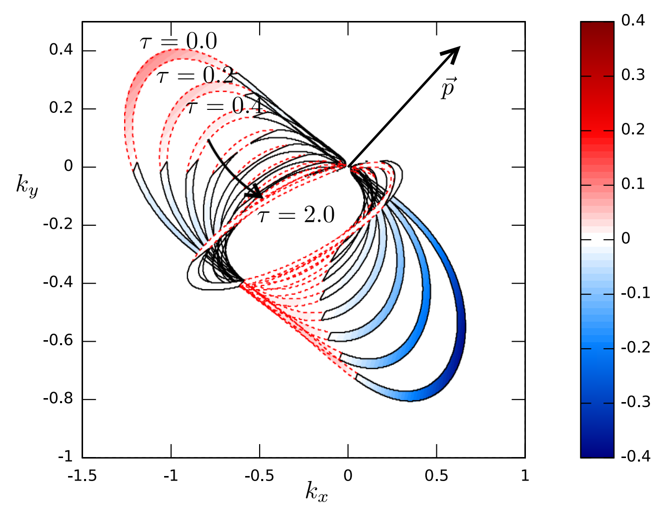

FIG. 2 (color online). The resonant manifold $\Delta\omega \equiv \omega(\mathbf{p}) + \omega(\mathbf{q}) + \omega(\mathbf{k}) \approx 0$, with $\mathbf{q} = -\mathbf{k} - \mathbf{p}$, in shearing coordinates as a function of time τ for $p_x = 0.6$, $p_y = 0.4$. The color intensity indicates the strength of the nonlinear interaction coefficient M_{kpq}, where positive and negative values are enclosed in dashed and solid lines, respectively. The figure shows that while initially this mode was coupled to two other modes roughly $k_x \sim -1$, $k_y \sim 0.3$ and $k_x \sim 0.6$, $k_y \sim -0.6$, later on, both the size of the resonant manifold and the nonlinear interaction coefficient on the resonant manifold diminish. This suggests that the three-wave interactions become very ineffective.

time increases, the eddies are tilted and they can only couple to the large scale flow.

Physically, this corresponds to tilting and absorbtion of small scale vortices by the sheared flow as discussed in Ref. [25]. This is a self-enhancing nonlinear mechanism akin to inverse cascade that amplifies the initial shear as opposed to the vortex breakup. This observation is interesting for control, because if we impose external flow shear, the same level of turbulence will generate more zonal flows under the action of large scale shear because the eddies cannot couple to anything else other than the zonal flows due to the destruction of the resonance manifold.

Flow shear also has an effect on the most unstable mode of the primary instability. Because we have argued that at a basic level, the effect of flow shear was to replace the linear growth rate γ, with an effective growth rate $\gamma_{\rm eff}$ in the original system. We can also compare the k_y dependencies of $\gamma_k = \gamma(k_y)$ and $\gamma_{{\rm eff},k} = \gamma_{\rm eff}(k_y)$ and check if the most unstable mode is modified.

For the weak shear case if $d\gamma_k/dk_y = 0$ at k_{y0}:

$$\frac{d\gamma_{\rm eff}}{dk_y}\bigg|_{k_y=k_{y0}} = -\frac{2}{3}\nu k_y\left(\gamma_k + \frac{\nu k_y^2}{3\gamma_k^2}v'^2\right)^{-2}.$$

Because we have a negative slope at the point of the old maximum, the maximum must have moved to smaller $|k_y|$. This means that the effect of flow shear is also to move the most unstable mode towards smaller k_y. This may seem somewhat counter intuitive, but it has to do with the fact that it takes more time for the flow shear to act on modes with smaller k_y (e.g., consider the limit $k_y \to 0$). The actual shift amount Δk_y depends on the form of γ_k. However, a small shift estimate can be given:

$$(k_{y0}^{\rm eff} - k_{y0}) = \frac{2\nu}{3}\frac{k_{y0}v'^2}{[\gamma^2 + 4\nu k_{y0}^2 v'^2/(3\gamma)]}\frac{1}{d^2\gamma/dk_y^2}$$

for weak values of shear, which is negative.

Some of the above results can be written in terms of a dimensionless variable $\mathcal{G} \equiv \nu^{1/3}k_y^{2/3}v'^{2/3}/\gamma$. Threshold for stabilization is given by the critical value $\mathcal{G}_c \approx 0.76$. Weak and strong shear solutions become $\gamma_{\rm eff} = \gamma f_1(\mathcal{G})$ and $\gamma_{\rm eff} = -\gamma_B f_2(\mathcal{G})$, respectively, where:

$$f_1(\xi) = \frac{1}{3}[A(\xi)^{1/3} + A(\xi)^{-1/3} + 1]$$

$$f_2(\xi) = \frac{3^{1/3}}{2\xi}\left(f_1(\xi) - 1 - \frac{i}{\sqrt{3}}[A(\xi)^{1/3} - A(\xi)^{-1/3}]\right)$$

$$A(\xi) = \left(3\xi\sqrt{\xi\left(\frac{9\xi^3}{4} - 1\right)} + 1 - \frac{9\xi^3}{2}\right)^{1/3}$$

and finally the shift of the most unstable mode can be written using the solution of the cubic equation as:

$$(k_{y0,{\rm eff}} - k_{y0}) = \frac{2\gamma}{3k_{y0}}\frac{1}{(1 - \frac{\mathcal{G}}{f_1(\mathcal{G})}\frac{df_1(\mathcal{G})}{d\mathcal{G}})}\frac{1}{d^2\gamma/dk_y^2}.$$

As conclusion, we have shown that a simple cubic equation defines the effective growth rate, which for weak shear goes as $\gamma_{\rm eff} = \gamma(1 + \gamma^{-3}\gamma_B^3)^{-1}$ and for strong shear becomes $\gamma_{\rm eff} = -\gamma_B(1 + \frac{\gamma}{3\gamma_B})^{-1}$. Using this, we have shown that the Hinton–Staebler shear-suppression formula used in various models of L-H transition can be seen as a mixing length estimate for the fluctuation level with a $\gamma_{\rm eff}$ given by the weak shear formula above. We have also noted that the weak shear solution disappears, and the strong shear solution becomes the only one available as the shear suppression threshold given by $|v'| = (2/3)\gamma^{3/2}/\sqrt{\nu k_y^2}$ is exceeded. We have also discussed the effect of shear on the mode coupling processes, using a dispersion relation and interaction coefficient for drift waves (applicable also to Rossby waves). We have shown that the direct three-wave mode coupling becomes more difficult as the shear tilts the eddies, as the resonant manifold is weakened by shear. This enhances the coupling with zonal flows as they satisfy the resonance trivially and are unaffected by shear. We have also pointed out that the shear causes the wave number of the most unstable mode to shift to lower k_y.

This work is supported by the French "Agence Nationale de la Recherche," Contract No. ANR JCJC 0403 01.

*ozgur.gurcan@cea.fr

[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[2] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University, Cambridge, England, 1995).
[3] P. H. Diamond, S.-I. Itoh, K. Itoh, and T. S. Hahm, Plasma Phys. Controlled Fusion **47**, R35 (2005).
[4] A. M. Balk, S. V. Nazarenko, and V. E. Zakharov, Phys. Lett. A **146**, 217 (1990).
[5] P. H. Diamond, Y.-M. Liang, B. A. Carreras, and P. W. Terry, Phys. Rev. Lett. **72**, 2565 (1994).
[6] I. N. James, J. Atmos. Sci. **44**, 3710 (1987).
[7] F. H. Busse and A. C. Or, J. Fluid Mech. **166**, 173 (1986).
[8] V. Berionni and Ö. D. Gürcan, Phys. Plasmas **18**, 112301 (2011).
[9] P. W. Terry, Rev. Mod. Phys. **72**, 109 (2000).
[10] K. H. Burrell, Phys. Plasmas **4**, 1499 (1997).
[11] B. B. Kadomtsev, Sov. J. Plasma Phys. **1**, 295 (1975).
[12] P. Goldreich and D. Lynden-Bell, Mon. Not. R. Astron. Soc. **130**, 125 (1965).
[13] H. Biglari, P. H. Diamond, and P. W. Terry, Phys. Fluids B **2**, 1 (1990).
[14] M. Lesieur, *Turbulence in Fluids* (Kluwer, Dordrecht, 1997), 3rd ed.

[15] F. L. Hinton and G. M. Staebler, Phys. Fluids B **5**, 1281 (1993).
[16] R. E. Waltz, G. D. Kerbel, and J. Milovich, Phys. Plasmas **1**, 2229 (1994).
[17] T. S. Hahm and K. H. Burrell, Phys. Plasmas **2**, 1648 (1995).
[18] C. Connaughton, S. Nazarenko, and B. Quinn, Europhys. Lett. **96**, 25 001 (2011).
[19] B. Dubrulle and S. Nazarenko, Physica (Amsterdam) **110D**, 123 (1997).
[20] A. I. Smolyakov and P. H. Diamond, Phys. Plasmas **6**, 4410 (1999).
[21] A. Hasegawa and K. Mima, Phys. Fluids **21**, 87 (1978).
[22] J. G. Charney, Geofys. Publ. Oslo **17**, 1 (1948).
[23] J. A. Krommes, Phys. Rep. **360**, 1 (2002).
[24] A. M. Balk, V. E. Zakharov, and S. V. Nazarenko, JETP Lett. **71**, 249 (1990).
[25] P. Manz, M. Ramisch, and U. Stroth, Phys. Rev. Lett. **103**, 165004 (2009).