Anisotropy in MHD turbulence with mean field: zero parallel cascade?

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This work deals with anisotropy of MHD (incompressible) turbulence with mean magnetic field. The anisotropy issue, in presence of a mean field is important as it determines the scaling and dissipation rate of turbulence and its scaling with $B^\circ$.

We use two different numerical tools to explore how the cascade occupies Fourier space during the nonlinear cascade.

PLAN

Introduction
(1) Full MHD, periodic boundaries, forcing in volume
(2) Reduced MHD with Shell model for: energy injection at open boundary
Introduction: why anisotropy?

Incompressible MHD equations with unit Prandtl number:
\[
\begin{align*}
\partial_t z^+ - B^0 \partial_x z^+ + (z^+. \nabla) z^+ + \nabla P &= R^{-1} \Delta z^+ + f^+ \\
\partial_t z^- + B^0 \partial_x z^- + (z^-+. \nabla) z^- + \nabla P &= R^{-1} \Delta z^- + f^-
\end{align*}
\]

\[
z^+ = u - b \quad z^- = u + b \quad (b = (B-B^0)/\sqrt{\rho^0}) \quad \text{div } z^+ = \text{div } z^- = 0
\]

Assume \textbf{B}^0 \textbf{l}\textbf{a}\textbf{r}\textbf{g}\textbf{e}\textbf{r} \textbf{t}\textbf{e}\textbf{r} \textbf{r}\textbf{h} \textbf{f} \textbf{l}\textbf{u}\textbf{c}t\textbf{u}\textbf{a}t\textbf{i}\textbf{o}\textbf{n} (b,u,z^\pm)

\begin{itemize}
  \item Gradients // to \textbf{B}^0 have strong \( B^0 \partial_x z^\pm \Rightarrow \text{NL term} << \text{LIN term} \)
  \item Gradients \( \perp \) to \textbf{B}^0 have ZERO \( B^0 \partial_x z \Rightarrow \text{NL term alone} \)
\end{itemize}

\textit{Isotropic case} \((B^0=0)\) \quad \textit{Anisotropic case} \((B^0 \text{\ large enough})\)
I Defining NL/LIN dominated regions

- NL time and LIN Alfvén propagation time
  \[ \tau_{NL} \approx ku, \quad \tau_A = kB^\circ \cos \theta = k_/B^\circ \]

- **Time ratio** \( X = \tau_A / \tau_{NL} \)

- Assuming \( u \approx l^{1/3} \approx k^{-1/3} \) (K41)

- NL/LIN boundary \( X = 1 \)
  \[ \cos \theta^* \approx k^{-1/3} \text{ or } k_/ \propto k_-^{2/3} \]

II Critical Balance conjecture

- Assume à la Kolmogorov cascade along \( k_- \)
- Assume **zero** cascade along \( k_/ \) (cf. Strauss equ., reduced MHD)

- **Conjecture 1:** \( // \) (linear) and \( \perp \) (NL) correlation times are equal:
  \[ \tau_/^{\text{cor}} = \tau_-^{\text{cor}} \quad (1) \]

- **Conjecture 2:** \( k_-^{5/3} \) spectrum fills region \( X \geq 1 \)
  \[ ... \text{ with negligible energy outside} \quad (2) \]
I Full MHD

1024^2 x 256 simulation results, \( \varepsilon = \frac{b_{\text{rms}}}{B^0} = 1/5 \)

- Continuous: 3D energy contours of 3D energy spectrum \( E_3(k, \theta) \)
- Dotted: energy contour of model 3D spectrum \( E_3(k, \theta) = \Lambda(\theta) k^{-m/2} \)
  \[ \Lambda(\theta) = (\varepsilon^{-2} \cos^3 \theta + \sin^3 \theta)^{-m/2} \]  
  \( m = 3/2 \)  

**Power-law identical in all directions**  
Inertial range extent varies with \( \theta \) as \( k_d(\theta) = k(\pi/2)(\varepsilon^{-2} \cos^3 \theta + \sin^3 \theta)^{-1/2} \)

\[ \Rightarrow k_d(0)/k_d(\pi/2) = \varepsilon = \frac{b_{\text{rms}}}{B^0} \]

- Excited region largely outside region \( X > 1 \)

**Contours of 3D energy spectrum, \( B^0 = 5b_{\text{rms}} \)**

**Contours of 3D energy spectrum, \( B^0 = 10b_{\text{rms}} \)**

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Anisotropy scaling with B°: quantifying // extent

1. Full MHD properties
Power-law extent given by dissipative wavenumber:
\[ k_d(\theta) = k(\pi/2)(\epsilon^{-\theta} \cos^3 \theta + \sin^3 \theta)^{-1/2} \]
- When B° grows (or \( \epsilon \) decreases):
  Max \( \perp \) extent : \( k_\perp \propto O(1) \)
  MAX // extent \( k_1 \propto \epsilon^{2/3} \)
  MIN // extent \( k_2 \propto \epsilon \)

2. Critical balance prediction
- Spectrum confined to \( X>1 \) region
  Energy spectrum as \( E_3(k,\theta)=f(X)k_\perp^{-m-2} \)
  with \( f(X) \) negligible when \( X<1 \).
  As B° grows (\( \epsilon \to 0 \))
  Max \( \perp \) extent : \( k_\perp \propto O(1) \)
  MAX // extent \( k_1 \propto \epsilon \)
  \( \Rightarrow \) CB spectrum becomes increasingly different from full MHD spectrum as B° grows

3. Conjecture for large B°
- Full MHD has
  - Reduced decrease of // spectral extent in oblique directions (\( \Delta k_{//} \approx \epsilon^{2/3} \))
  - Reduced dissipation compared to Kolmogorov perpendicular cascade scaling as \( t_{NL}^{-1} \epsilon^{2/3} \)
Radial cascade?

• In contrast with the isotropic forced scales, the scales which play the role of the large scale input \((k=k^* \approx 8)\) are NOT isotropic: they follow a definite \(A(\theta)\) dependence which depends on \(\epsilon\).
• Hence, one can define an effective Reynolds at each angle \(\theta\), which controls a (possibly) radial cascade starting with \(k=k^*\).

Power-law range extent

1. Hyp. of pure radial cascade + standard argument on large scale input \((k^*)\), small-scale sink \((k_d)\), and Reynolds \(Re \approx A(\theta)^{1/2}\):
\[
k^* u^3 \approx \nu k^2 d^2, \quad u_k^2 \approx kE(k,\theta) \approx A(\theta) k^{-1/2}
\]

\[
k_d \approx \text{Re}^{2/3} \quad (1)
\]

2. Simulation gives \(E_3(k_d,\theta) = \text{const} = A(\theta) k_d^{-7/2}\)

\[
k_d \approx \text{Re}^{4/7} \quad (2)
\]

Conclusion

• power-law range extent less anisotropic than would predict purely radial en. flux
• radial range NOT inertial, \(\perp\) scales directly feed \(\parallel\) scales

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Interplay of waves and NL coupling

**Residual energy** \( E_R(k) = E_M(k) - E_V(k) \) related to **total energy** \( E_T(k) = E_M(k) + E_V(k) \)

\[
\partial_t E_R(k) = -E_R(k)/t_A + E_T(k)/t^*
\]

\( t_A = \) isotropized Alfvén time \( = (kB^o)^{-1} \), \( t^* = t_{NL} (b/B^o) \) (IK long interaction time)

Equilibrium state:

\[
E_R = (t_A/t_{NL})^2 E_T \quad \Rightarrow \quad E_R \approx k E_T^2
\]

- 2 cases: \( E_T \approx k^{-5/3} \) \( \Rightarrow \) \( E_R \approx k^{-7/3} \); \( E_T \approx k^{-3/2} \) \( \Rightarrow \) \( E_R \approx k^{-2} \)

- Indicates quasi-// Alfvén waves play a more active role than just propagating \( \perp \) fluctuations along mean field

\[\text{Residual energy spectrum norm. by } k^{-7/3}\]

\[\text{Residual } \perp \text{ energy spectrum norm. by } k^{-2}\]

\( B^o = 5b_{\text{rms}}, \ E_T \approx k^{-3/2} \)

\( \text{NO mean field, } E_T \approx k^{-5/3} \)
II Reduced MHD - Shell model

• Reduced MHD
  \[ \text{div } z^\pm = 0 \]
  \[ \partial_t z^\pm + B^\circ \partial_x z^\pm + NL^\pm = R^{-1} \Delta z^\pm + F^\pm \]

  but with \[ \nabla = (0, \partial_y, \partial_z) \] in \( NL^\pm \)

• SHELL model:
  Replace NL terms by shell model (as many shell models as grid points along x-axis)

NB Shell model increase Reynolds with Log discretization of wave numbers
here, 20 shells \(\Rightarrow 2^{20} \approx 10^6\) mesh points in the two directions of the \( \perp \) plane...

Forcing \( z^- \)

Forcing \( z^+ \)

\[ 0 \]
\[ L \]
\[ 0 \]
\[ L \]
\[ 0 \]
\[ L \]

\[ \text{perpendicular cascade} \]

\[ \text{parallel propagation} \]

Spectra $E^{\pm}(x,k)$

$\text{brms}/B^o = 0.05$; $L_\perp = 2\pi/k^o = 0.01 L$ (aspect ratio 100)

$\perp$ modes $k_n$ ($n=2,3,4$) injected at $x=0$ and $L$

When $z^+$ and $z^-$ modes meet, they start a cascade to larger $k_\perp$ (larger $n$).

$=>$ spectrum $E^{\pm}(k_n,x)$, different at each point $x$

- Forcing such as $t_{NL} \ll t_A = L/V_A$: spectra have time to develop before leaving domain

**Time-averaged spectra** $k^{5/3} E^{\pm}(x,k)$

**Snapshot of spectra**

1. Time-averaged spectra (7 Alfvén times): normalizing by $k^{-5/3}$ reveals $5/3$ slope except close to boundaries ($x=0, L$)

2. Snapshots reveal $//$ structures which decrease at small $\perp$ scales

$=>$ Test for critical balance and anisotropy?
Critical balance between perp and // times?

Each perpendicular mode in the domain \([0,L] \times [1,N]\) is a random signal

One may define for each mode at each point \(x\) of the domain:

- a correlation time \(T_{\text{cor}}(n)\) and average on \(x\)
- a nonlinear time based on amplitude of \(u_n\) and wavenumber \(k_n\) and average on \(x\)
- a parallel correlation length

\[ T_{\text{cor}}, L_{\text{cor}}/B^\circ, t_{\text{NL}} \]

For each \(n\), compute characteristic times \((n=\log_2(k)):\)

1. \(\perp\) correlation time \(T_{\text{cor}}(n)\)
2. // correlation length \(L_{\text{cor}}(n)\) and // correlation time \(L_{\text{cor}}/B^\circ\)
3. NL time \(t_{\text{NL}}(n) = (k_n | u_n |)^{-1}\)

In inertial range, all three time scales coincide:

\(\Rightarrow\) Shell model show critical balance of NL perp and linear // times
(preliminary) Full angular spectra $E(k_\parallel, k_\perp)$: anisotropy

- Spectra $E^{\pm}(x, k)$ **transformed into full spectra** $E^{\pm}(k_\parallel, k_\perp)$ via FFT along x axis
- NL forcing time $<< L/V_A$

NB Using cartesian-Log coordinates spoils anisotropy in Fourier space $E^{\pm}(\log|k|)$ in each direction $\theta$ should be used instead, as below

NB2 Needs more checking: normalization AND interpolation (very large aspect ratio)

(Polar) angular energy spectrum $E(\log|k|, \theta)$

$1D$ energy spectra along various directions

\[ X=1 \]

Log $k_\perp$

Log $k_\parallel$

forcing
Summary

1. Full MHD, periodic boundaries, volume forcing
Properties distinct from critical balance predictions are found when $B^° > b_{\text{rms}}$
A uniform 1D spectral slope is found in all directions; anisotropy has a weak scaling with $\varepsilon = b_{\text{rms}}/B^°$, dissipation is reduced, varying as $\varepsilon^{2/3}$. This is valid for:
   • power-law range as well as dissipative range
   • resistive and ideal runs (1D slope resp. $m=3/2$ and $m=5/3$)
   • Reynolds numbers from 100 to 1000
   • $B^°/b_{\text{rms}} = 5$ and 10

2. Reduced MHD, injection through open boundaries
This model adopts from start zero nonlinear parallel interactions
It thus allows to explore in detail the basic predictions of the critical balance (CB) paradigm

   • Each scale of the perpendicular inertial range ($5/3$ power-law) have the 3 time scales equal: (i) the perpendicular correlation time (ii) the nonlinear time (defined as $k |u|^{-1}$)
     (iii) the linear propagation time (defined as $L_{\text{cor}}/V_A$)

   • However, anisotropy of energy spectrum $E(k_//, k_\perp)$ might be at odd with CB predictions